ALMOST-FACTORIAL QUARTIC SURFACES WITH A TACNODAL POINT IN $\mathbb{P}^3$

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Abstract

The behaviour of birational transformations between particular surfaces in $\mathbb{P}^3$ and almost-factoriality are investigated. Using a suitable parametrization on $\mathbb{P}^2$ of the quartic surfaces with a tacnodal point, all almost-factorial surfaces of this kind can be classified. We prove that there are nine classes of such surfaces $\mathcal{F}$, and for each of them a possible equation is written and its index of almost-factoriality $\nu$ is computed. There are surfaces $\mathcal{F}$ with $\nu = 4, 8, 12$. For each irreducible algebraic curve $\mathcal{C} \subset \mathcal{F}$, we outline how to construct a surface $\mathcal{G}$ such that $\mathcal{F} \cdot \mathcal{G} = \mu \mathcal{C}$, with $\mu \leq \nu$.

1. Introduction

There are classes of algebraic surfaces $\mathcal{F} \subset \mathbb{P}^3$ that have the following property: For every algebraic curve $\mathcal{C} \subset \mathcal{F}$, there is an algebraic surface $\mathcal{S}$ such that $\mathcal{F} \cap \mathcal{S} = \mathcal{C}$, or, more precisely, such that $\mathcal{F} \cdot \mathcal{S} = \mu \mathcal{C}$, where $\mu$ is the multiplicity $\mu = I(\mathcal{C}, \mathcal{S} \cap \mathcal{F})$ of intersection.
between $\mathcal{F}$ and $\mathcal{S}$ along $\mathcal{C}$. Such a surface $\mathcal{F}$ is called a \textit{set-theoretic complete intersection surface}; and if $\mathcal{F}$ is non-singular in codimension 1, then it is called \textit{almost-factorial (factorial if $\mu = 1$)}.

We know that the \textit{index of almost-factoriality} of $\mathcal{F}$ is the integer number $\nu$ such that, for every reduced and irreducible curve $\mathcal{C} \subset \mathcal{F}$, there is a surface $\mathcal{S}$ with the property that $\mathcal{S} \cdot \mathcal{F} = \mu \mathcal{C}$, where $\mu \leq \nu$. In this case, we say that $\mathcal{F}$ is $\nu$-\textit{almost-factorial}.

It is well known that the multiplicity of intersection $\mu$ between two surfaces $\mathcal{F}$ and $\mathcal{S}$ along a reduced and irreducible curve $\mathcal{C}$ can be computed by considering their affine parts in a suitable affine space $\mathbb{A}^3$ in $\mathbb{P}^3$, such that $\mathcal{C}_a : \mathcal{C} \cap \mathbb{A}^3$ is a curve again. If one of such surfaces $\mathcal{F}$ is normal, the following valuation can be used. Let $\mathcal{F}_a : F = 0$, $\mathcal{G}_a : G = 0$, and $\mathcal{C}_a$ be the affine parts of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{C}$, with $F, G \in k[X, Y, Z]$. Let $k[\mathcal{F}_a] = k[X, Y, Z]/(F) = k[x, y, z]$ and $K = k(\mathcal{F}_a)$ be the quotient field of $k[\mathcal{F}]$, where $x, y, z, g$ denote the canonical projections of the polynomials $X, Y, Z, G$ on $k[\mathcal{F}_a]$. Let $\mathfrak{p}$ be the prime ideal of $\mathcal{C}_a$ in $k[\mathcal{F}_a]$, and $v$ be the valuation centered at $\mathfrak{p}$ of the local ring D.V.R. $k[\mathcal{F}_a]_{\mathfrak{p}}$. It will defined $\mu = v(g)$.

Many classes of almost-factorial surfaces in $\mathbb{P}^3$ are well known. It is worth recalling them here.

The planes are factorial. Only the quadrics $\mathcal{F}_2$ with a unique double point (cones) are 2-almost-factorial. Among the cubic surfaces $\mathcal{F}_3$, there are only three families of almost-factorial surfaces $\mathcal{F}_3 \subset \mathbb{P}^3$ (see [13]). Every irreducible quadric cone or cylinder in $\mathbb{A}^3$ is 2-almost-factorial, and every quadric paraboloid in $\mathbb{A}^3$ is factorial. For the affine cubic surfaces $\mathcal{F} \subset \mathbb{A}^3$, there are 82 families of factorial or almost-factorial surfaces (see [4]).
For quartic surfaces $\mathcal{F}_4 \subset \mathbb{P}^3$, we know that the “generic” non-singular surface is factorial (according to Nöther’s theorem generalized first by Gröbner; Andreotti and Salmon; and later by Deligne, see [8], [1], [5]), and 33 families of almost-factorial quartic monoids in $\mathbb{P}^3$ have been described (see [11], [12]).

The normal quartic, which is a Zariski’s surface $Z_p : Z^p = G_4$

$(X, Y) \subset \mathbb{A}^3_k$, where $k$ is an algebraically closed field of characteristic $p > 0$ and $G_4$ is a polynomial of degree $\leq 4$, was examined particularly in a book by Lang (see [3]), in which (pages 150-171) the author tackles the factoriality or almost-factoriality of these surfaces.

Biregular birational transformations between algebraic varieties are known to preserve their almost-factoriality (see [2]). A criterion has been given in [6] for the almost-factoriality of $V$ when a birational transformation of $\mathbb{P}^n$ is encountered in a projectively normal variety $V$.

To the best of our knowledge, nobody knows whether any quartic surfaces in $\mathbb{P}^3$ with only double points on them are factorial or almost-factorial.

The aim of this paper was to exhaustively answer the question of which normal quartic surfaces in $\mathbb{P}^3$ with a tacnodal point on them are almost-factorial.

All the almost-factorial quartic surfaces in $\mathbb{P}^3$ can be placed in 9 classes to within a linear change of coordinates. Using suitable equations for these surfaces $\mathcal{F}$, we adopt a constructive process to obtain a surface $\mathcal{G}$ such that for every curve $\mathcal{C}$ on $\mathcal{F}$, we shall have $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$.

To solve the proposed problem, it is essential to analyze the birational transformations of the surfaces in $\mathbb{P}^3$. In the following paragraphs, $k$ denotes an algebraically closed field of characteristic
\( p = 0, \) and \( A^3 \) and \( P^3 \) are the affine and projective spaces on \( k \) of dimension 3. A point \((a, b, c) \in A^3\) will also be identified with \((1 : a : b : c) \in P^3\).

### 2. Almost-Factoriality of Surfaces in \( P^3 \) and Birational Transformations

Let \( F = F(T, X, Y, Z) \) and \( G = G(T_1, X_1, Y_1, Z_1) \) be irreducible homogeneous polynomials, and the surfaces \( \mathcal{F} : F = 0 \subset P^3 \) and \( \mathcal{G} : G = 0 \subset P^3_1 \) be non-singular in codimension 1. Then, let be

\[
\begin{align*}
  k[\mathcal{F}] &= k[T, X, Y, Z] / J(\mathcal{F}), \\
  k[\mathcal{G}] &= k[T_1, X_1, Y_1, Z_1] / J(\mathcal{G}).
\end{align*}
\]

These rings can be regarded as the rings of regular functions of the affine cones over \( \mathcal{F} \) and \( \mathcal{G} \). They are both integral closed rings because \( \mathcal{F} \) and \( \mathcal{G} \) are non-singular in codim 1. We must remember that a surface \( \mathcal{F} \subset P^3 \) is normal, if it is non-singular in codim 1 and, as a complete intersection of \( P^3 \), it is projectively normal to (see [9], Example 84.5, page 188).

Every rational transformation between projective surfaces in \( P^3 \) can be regarded as the restriction of suitable transformations of projective space.

Let us consider the following rational transformation: \( \tau : P^3 \longrightarrow P^3_1 \)

\[
\tau : (T_1 : X_1 : Y_1 : Z_1) = (H_0 : H_1 : H_2 : H_3),
\]

given by the four homogeneous polynomials of the same degree

\[
H_i \in k[T, X, Y, Z], \quad i = 0, \ldots, 3,
\]

whose remainders mod \( J(\mathcal{F}) \) does not have a common factor; and the transformation \( \sigma : P^3_1 \longrightarrow P^3 \)
\[ \sigma : (T : X : Y : Z) = (L_0 : L_1 : L_2 : L_3), \]
given by the four homogeneous polynomials of the same degree
\[ L_i \in k[T_1, X_1, Y_1, Z_1], \quad i = 0, \ldots, 3, \]
whose remainders \( \text{mod } J(\mathcal{G}) \) does not have a common factor. We also have a birational transformation from \( \mathcal{F} \) to \( \mathcal{G} \), if the following holds:
\[ F(H_0, \ldots, H_3) \in J(\mathcal{F}), \quad G(L_0, \ldots, L_3) \in J(\mathcal{G}), \quad (1) \]
and there are two non-vanishing polynomials \( N \in k[T, X, Y, Z] \) and \( N_1 \in k[T_1, X_1, Y_1, Z_1] \), for which the following holds:
\[ \sigma \circ \tau : (T_1 : X_1 : Y_1 : Z_1) = (N_1T_1 : N_1X_1 : N_1Y_1 : N_1Z_1) \text{ mod } J(\mathcal{G}), \quad (2) \]
and
\[ \tau \circ \sigma : (T : X : Y : Z) = (NT : NX : NY : NZ) \text{ mod } J(\mathcal{F}). \quad (3) \]

The relations (1), (2), (3) imply that the restrictions of \( \tau \) on \( \mathcal{F} \cap \{N \neq 0\} \) and of \( \sigma \) on \( \mathcal{G} \cap \{N_1 \neq 0\} \) are regular maps, one being the inverse of the other.

Indeed, if \( P = (t_P : x_P : y_P : z_P) \in \mathcal{F} \cap \{N \neq 0\} \) from (2), then we shall have
\[ P = (N(P)t_P : N(P)x_P : N(P)y_P : N(P)z_P) = \sigma(\tau(P)), \]
with \( \tau(P) = (L_0(P) : L_1(P) : L_2(P) : L_3(P)) \), and with \( L_i(P) \neq 0 \), for at least one \( i, i = 0, \ldots, 3 \) (otherwise \( \sigma((0 : 0 : 0 : 0)) = (0 : 0 : 0 : 0) \) is not the point \( P \)). By this and from (1), we obtain \( \tau(P) \in \mathcal{G} \).

Below, we call \( \mathcal{F} \cap \{N \neq 0\} \) the set of regularity of \( \tau \).

In the same way, we can see that, if \( Q \in \mathcal{G} \cap \{N_1 \neq 0\} \), the point \( \sigma(Q) \) exists and \( \sigma(Q) \in \mathcal{F} \).
Proposition 1. Let $\tau : \mathcal{F} \to \mathcal{G}$ be a birational transformation between two normal algebraic surfaces in $\mathbb{P}^3$, and let $\mathcal{G}$ be almost-factorial. $\mathcal{F}$ is almost factorial if and only if every irreducible curve $\mathcal{D} \subset \mathcal{F}$, whose image is a point on $\mathcal{G}$ is a set-theoretic complete intersection of $\mathcal{F}$.

Proof. For $\tau$ we keep the previously notations. The assumption that $\dim \mathcal{F} = \dim \mathcal{G} = 2$ implies that $\tau$ gives a birational transformation that induces a $k$ isomorphism $\tau^*$ between the rational fields on $\mathcal{G}$ and $\mathcal{F}$. We can apply “Zariski’s main theorem” (see [10], page 49 in the form given by Bourbaki, Chapter 5, Examples 4-7) to the restriction of the birational transformation $\tau$ between the affine varieties $X^2 = \mathcal{F} \cap \{N \neq 0\}$ and $Y^2 = \mathcal{G} \cap \{N_1 \neq 0\}$ of $\mathbb{P}^3$ because $X^2$ and $Y^2$ are non-singular in codim 1 (so they are projectively normal). According to this theorem, for said $\gamma = \tau(x)$ to be a point at $Y^2$ for $x \in X^2$ can only happen in one of two situations:

1. if $\tau^{-1}$ is regular at $y$, or
2. if there is a divisor $\mathcal{D}$ on $\mathcal{F}$, $x \in \mathcal{D}$ (called an exceptional divisor), the projective closure $\overline{\tau(\mathcal{D})}$ has dimension 0.

An irreducible curve on $\mathcal{F} \cap \{N \neq 0\}$ can therefore be the pre-image either of a curve or of a point on $\mathcal{G} \cap \{N_1 \neq 0\}$, so the hypothesis in Proposition 1 is necessary. Now, we have to demonstrate that it is also sufficient.

According to the hypothesis in Proposition 1, every irreducible curve $\mathcal{D}'$ on the set of non-regularity for $\tau$ is a set-theoretic complete intersection of $\mathcal{F}$ with a surface $\mathcal{R}'$ in $\mathbb{P}^3$. To show that $\mathcal{F}$ is almost-factorial, we need to verify that every irreducible curve $\mathcal{D}$ on $\mathcal{F} \cap \{N \neq 0\}$, whose image is a curve on $\mathcal{G}$, $\mathcal{C} = \tau(\mathcal{D})$ is actually a set-theoretic complete intersection of $\mathcal{F}$. 
Denoting the restrictions of $\tau$ as $\tau^*, \tau'$, we have the following situation:

$$D \subset F \subset P^3$$

$$\downarrow \tau^* \quad \downarrow \tau' \quad \downarrow \tau.$$

$$C \subset G \subset P^3_1$$

As $G$ is almost-factorial, a surface $\mathcal{H} : H(T_1, X_1, Y_1, Z_1) = 0$ in $P^3_1$ exists such that $\mathcal{H} \cap G = C = \tau(D)$. Let us consider the polynomial $H(H_0, H_1, H_2, H_3) = L(T, X, Y, Z)$ and denote the surface $\mathcal{L} : L(T, X, Y, Z) = 0$.

We consider the divisor

$$F \cdot L = vD + v_1D_1' + ... + v_tD_t', \quad v > 0, v_i > 0, 1 \leq j \leq t.$$ 

None of the components $D_j' \neq D, 1 \leq j \leq t$, can be transformed in $C$ because $\tau$ is invertible on $C = \tau(D)$, so it belongs to $F \cap \{N = 0\}$, and $\tau(D_j')$ is a point on $G$.

Based on the hypothesis in Proposition 1, there are suitable surfaces $\mathcal{R}_j : R_j(T, X, Y, Z) = 0, 1 \leq j \leq t$, that cut on $F$ the divisors

$$\mathcal{R}_j \cdot F = \mu_j D_j', \quad \mu_j > 0, \quad 1 \leq j \leq t.$$

Let $\mu = m.c.m.\{\mu_1, ..., \mu_t\}$ and $R = \prod_{j=1}^{\mu} R_j^{\nu_j}$. If we consider the surface $\mathcal{R} : R = 0$ in $P^3$, we have first

$$F \cdot \mathcal{R} = \mu(v_1D_1' + ... + v_tD_t'),$$

and afterwards the canonical projections of $L$ and of $R$ in $k[\mathcal{F}]$, called $l$ and $r$, provide

$$\text{div}(\frac{\mu}{r}) = \mu(vD + v_1D_1' + ... + v_tD_t') - \mu(v_1D_1' + ... + v_tD_t') = \mu(vD).$$
We can apply the theorem of the integral closed Noetherian’s domain to $k[\mathcal{F}]$, and this enables us to find a polynomial $S \in k[T, X, Y, Z]$ such that it defines the divisor $\mu \nu \mathcal{D}$ on $\mathcal{F}$. The surface $S : S = 0$, thus intersects $\mathcal{F}$ along the curve $\mathcal{D}$ with multiplicity $\mu \nu$. $\mathcal{D}$ is therefore a set-theoretic complete intersection of $\mathcal{F}$, and $\mathcal{F}$ is therefore almost-factorial.

**Lemma 1.** If a pair of skew curves $\mathcal{C}_1$ and $\mathcal{C}_2$ exists on a surface $\mathcal{F}$ of $\mathbb{P}^3$, then $\mathcal{F}$ is not almost-factorial.

**Proof.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two skew curves on $\mathcal{F}$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. If $\mathcal{F}$ is almost-factorial, then there are two surfaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{P}^3$ such that

$$\mathcal{F} \cdot \mathcal{H}_1 = n_1 \mathcal{C}_1, \quad \mathcal{F} \cdot \mathcal{H}_2 = n_2 \mathcal{C}_2.$$ 

As every curve in $\mathbb{P}^3$ has a non-empty intersection with every surface, we have the contradiction

$$\emptyset \neq \mathcal{C}_1 \cap \mathcal{H}_2 = \mathcal{F} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{C}_1 \cap \mathcal{C}_2.$$ 

**Remark 1.** The proof of Proposition 1 enables us to construct a suitable surface $S$ for which, given an almost-factorial surface $\mathcal{F}$ and an irreducible curve thereon $\mathcal{D}$, it holds that $\mathcal{D} = \mathcal{F} \cap S$. In the case of $\mathcal{F}$ being rational, see the example in [6].

**Lemma 2.** Let $\mathcal{F}$ be a normal surface in $\mathbb{A}^3$, $\deg \mathcal{F} = n \geq 2$ and $\mathbf{r}$ be a straight line on $\mathcal{F}$. For $\mathbf{r}$ to be a complete intersection of $\mathcal{F}$ with a surface $\mathcal{G}$,

$$\mathcal{F} \cdot \mathcal{G} = \mu \mathbf{r}, \quad \mu = \deg \mathcal{F} \deg \mathcal{G},$$

it is necessary that the plane tangent to $\mathcal{F}$ along $\mathbf{r}$ remains fixed.

**Proof.** This follows from the statement proved in [7].
3. The Quartic Surfaces in $\mathbb{P}^3$ with a Tacnodal Point

Below, $\mathcal{F}$ will be a normal quartic algebraic surface in $\mathbb{P}^3$ with a tacnodal point, that we can assume to be $(0 : 0 : 0 : 1) = Z_{\infty}$, and we can take the cone tangent to $\mathcal{F}$ at it (such a cone consists of two coincident planes) to be $T = 0$ (called the tacnodal tangent plane). $\mathcal{F}$ will then be given by the equation:

$$Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0, \quad \phi_4, \phi_2 \in k[T, X, Y],$$

where $\phi_4$ and $\phi_2$ are homogeneous polynomials, $\phi_4$ of degree four and $\phi_2$ of degree two, or $\phi_2 = 0$. Let us denote $\Delta = \phi_2^2 - \phi_4$, then the surface $\mathcal{F}$ can be represented again with an equation in the form

$$\mathcal{F} : (ZT + \phi_2)^2 - \Delta = 0.$$  

We call $T = 0$, the plane to infinity of the affine space $\mathbb{A}^3 = \mathbb{P}^3 \setminus \{T = 0\}$.

To recognize and classify the kinds of quartic almost-factorial surfaces with a tacnodal point, we consider the following obvious facts: The section between $\mathcal{F}$ and its tacnodal tangent plane $T = 0$, i.e., $\mathcal{F}_{\infty} = \mathcal{F} \cap \{T = 0\}$, is splitting into no more than four distinct lines; the cone $\Delta = 0$ of vertex $Z_{\infty}$ is invariant under linear transformation of coordinates in $\mathbb{A}^3 = \mathbb{P}^3 \setminus \{T = 0\}$, leaving the plane $T = 0$ and the point $Z_{\infty}$ unchanged.

**Lemma 3.** Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ be a quartic surface in $\mathbb{P}^3$. Then

1. if two linear polynomials $\phi_1 = aX + bY + cT, \psi_1 = a'X + b'Y + c'T$, exist with $ab' - a'b \neq 0$, such that both divide $\phi_4$, then a pair of skew lines exists on $\mathcal{F}$;
(2) if a linear polynomial $\phi_1 = aX + bY + cT$, divides $\phi_2$, and $\phi_1^2$ divides $\phi_4$, then $\mathcal{F}$ is singular along the line $\{T = \phi_1 = 0\}$;

(3) a surface $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ has tacnodal points at $O$ and at $Z_\infty$, if and only if the polynomials $\phi_2, \phi_4$ are in $k[X, Y]$;

(4) if a surface $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ has tacnodal points at $O$ and at $Z_\infty$ and their tangent tacnodal planes intersect at least in two distinct points $A_\infty$ and $B_\infty$ of $\mathcal{F}_\infty$, then $\mathcal{F}$ has the skew lines $Z_\infty A_\infty$, $OB_\infty$;

(5) on the surface $\mathcal{F} : Z^2T^2 + 2ZT(aX^2 + 2bXY + cY^2) + X^4 = 0$ if it is $c = 0$, then $\mathcal{F}$ is singular along the line $\{X = T = 0\}$. Assuming that $c = 1$, to within a substitution $Y = -bX + Y_1$, we can rewrite the equation for $\mathcal{F}$ in the form $Z^2T^2 + 2ZT[(a - b^2)X^2 + Y_1^2] + X^4 = 0$. For $a = b^2 \pm 1$, $\mathcal{F}$ is singular along the curves $\{Y_1 = 0 = ZT \pm X^2\}$.

Proof. (1) The lines $\{Z = 0 = aX + bY + cT\}$ and $\{T = 0 = a'X + b'Y + c'T\}$ exist on $\mathcal{F}$. From $ab' - a'b \neq 0$, their intersection is $\{T = Z = X = Y = 0\} = 0$.

(2) The section of $\mathcal{F}$ and the pencil of the planes $\{\phi_1 - \lambda T = 0\}$ is given by

$$Z^2T^2 + 2ZT\lambda T\phi_1' + (\lambda T)^2\phi_2' = T^2(2Z^2 + 2Z\phi_1' + \lambda^2\phi_2') = 0,$$

with suitable $\phi_1', \phi_2' \in k[T, X, Y]$.

We obtain the line $\{T = 0 = \phi_1\}$ counted twice for every $\lambda \in k$. $\mathcal{F}$ is then singular along such a line.

(3) $\mathcal{F} : Z^2T^2 + 2ZT\phi_2(X, Y) + \phi_4(X, Y) = 0$ remains invariable under the symmetry of $\mathbb{P}^3$, which changes $Z$ with $T$. The symmetric of the tacnodal point $Z_\infty$ is the point $O$, which is then a tacnodal point on $\mathcal{F}$,
and the plane $Z = 0$ is the tacnodal plane tangent to $\mathcal{F}$ at $O$. Conversely, if $O$ is another tacnodal point of $\mathcal{F}$, we can assume that the tangent tacnodal plane will be $Z = 0$. The equation for $\mathcal{F}$ will be unchanged.

(4) $Z_{\alpha}A_{\alpha} \cap OB_{\alpha} = 0$.

(5) If $c = 0$, then $X$ divides $aX^2 + 2bXY$ and $X^2$ divides $X^4$. Then, from (2), $\mathcal{F}$ is singular along \{X = T = 0\}. We can assume that $c = 1$.

Substituting $Y = -bX + Y_1$ in $Z^2T^2 + 2ZT(aX^2 + 2bXY + Y^2) + X^4 = 0$, we obtain $Z^2T^2 + 2ZT[(a - b^2)X^2 + Y_1^2] + X^4 = 0$; if $a - b^2 = \pm 1$, then $\mathcal{F} : (ZT \pm X^2)^2 + 2ZTY_1^2 = 0$. At every point on the curves \{Y_1 = 0 = ZT \pm X^2\}, the four partial derivatives of $(ZT \pm X^2)^2 + 2ZTY_1^2$ become zero. This shows that $\mathcal{F}$ is singular along these curves.

**Proposition 2.** A quartic surface in $\mathbb{P}^3_k$ that is non-singular in codimension 1 with two tacnodal points is not almost-factorial, if the field $k$ is supposed more than numerable.

**Proof.** Based on Lemma 3, points (4) and (5), we can assume that

$$\mathcal{F} : Z^2T^2 + 2ZT(aX^2 + Y^2) + X^4 = 0, \quad a^2 \neq 1.$$ 

Let $\sigma : (T_1 : X_1 : Y_1 : Z_1) = (TX^2 : X^3 : TXY : T^2Z)$ be the composition of two blow-ups, one centered in the tacnodal point $O$ of $\mathcal{F}$, and one on a line infinitely near to $O$. The restriction on $\mathcal{F}$ of $\sigma$ has as inverse the rational transformation

$$\sigma^{-1} : (T : X : Y : Z) = (T_1^3 : T_1^2X_1 : T_1X_1Y_1 : X_1^2Z_1).$$

The proper transform by $\sigma$ of $\mathcal{F}$ is the cubic surface

$$\mathcal{G} : Z_1^2T_1 + 2Z_1(aT_1^2 + Y_1^2) + T_1^3 = 0,$$
\( G \) is a non-singular cone in codimension 1 (elliptic), if \( a^2 \neq 1 \). So \( G \) and \( F \) are birationally equivalent non-rational surfaces. The locus of the non-regularity of \( \sigma \) is given by \( M = 0 \), where \( M = T^2X^6 \), because
\[
\sigma \circ \sigma^{-1} : (T : X : Y : Z) = (T^3X^6 : T^2X^7 : T^2X^6Y : T^2X^6Z).
\]
The section of \( F \) with the set of non-regularity of \( \sigma \) is
\[
F \cap \{XT = 0\} = \{X = Z = 0\} \cup \{X = T = 0\} \cup \{X = ZT + 2Y^2 = 0\}.
\]
The lines \( r : \{X = Z = 0\}, s : \{X = T = 0\} \) are the complete sections of \( F \) with the tacnodal planes tangent to \( F \) at \( O \) and at \( Z_\infty \).

Now, we show that the conic \( \mathcal{C} : \{X = ZT + 2Y^2 = 0\} \) is a set-theoretic complete intersection of \( F \). \( F \) is normal, so \( \text{div} \left( \frac{-x^4}{zt} \right) \) on it is
\[
4C + 4r + 4s - 4r - 4s = 4C = \text{div} \left( \frac{zt^2 + 2zt(ax^2 + y^2)}{zt} \right)
= \text{div} (zt + 2(ax^2 + y^2)).
\]
The surface \( \{ZT + 2(aX^2 + Y^2) = 0\} \) thus intersects \( F \) along \( 4C \).

So, from Proposition 1, \( F \) is almost-factorial if and only if \( G \) is almost-factorial.

We know that \( G \) is not almost-factorial (see [13], Proposition 11, page 171, where the field \( k \) is supposed more than numerable), so \( F \) will not be either.

**Lemma 4.** Let \( \mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0, \phi_4, \phi_2 \in k[T, X, Y], \) be a normal quartic almost-factorial surface. Then, the plane tangent to \( \mathcal{F} \) along every component line of \( \mathcal{F}_{x} \) can only be \( T = 0 \).

In addition, only one of the following cases can happen:

1. \( \mathcal{F}_{x} = 2r + 2s, r, s \) straight lines, \( r \neq s; \)
(2) $\mathcal{F}_\infty = 4r$.

If $T$ divides $\phi_2$, then only case (2) holds.

**Proof.** Let $r \cup s \subset \mathcal{F}_\infty$ with $r \neq s$. We can assume, for example, that $r : T = Y = 0$, $s : T = X + aY = 0$, $a \in k$. From Lemma 2, it follows that the plane tangent to $\mathcal{F}$ along $r$ remains fixed; if it were different from $T = 0$, let us assume, for instance, that $\alpha : Y = 0$. From $I(r, \mathcal{F} \cap \alpha) \geq 2$, there must therefore be suitable homogeneous polynomials $\phi_1', \phi_3', \phi_2' \in k[T, X, Y]$, such that, $\phi_2 = Y\phi_1' + T\phi_1$, $\phi_4 = Y\phi_3' + T^2\phi_2'$, resulting in

$$\mathcal{F} : T^2(Z^2 + 2Z\phi_1' + \phi_2') + Y(2ZT\phi_1' + \phi_3') = 0.$$ 

The section between $\mathcal{F}$ and $\alpha$ is

$$\mathcal{F} : \{Y = 0\} = 2r + C,$$

where $C : \{Y = Z^2 + 2Z\phi_1' + \phi_2' = 0\}$.

The two curves $s$ and $C$ on $\mathcal{F}$ are skew because their intersection is

$$s \cap C = \{T = X + aY = Y = Z^2 + 2Z\phi_1' + \phi_2' = 0\} = 0.$$ 

From Lemma 1, this contradicts the assumption that $\mathcal{F}$ is almost-factorial.

$\mathcal{F}_\infty$ thus consists of two distinct lines at most. Along these lines, the (fixed) plane tangent to $\mathcal{F}$ is $T = 0$. So we can only have the two situations

$$\mathcal{F}_\infty = 2r + 2s, \quad r, s \text{ straight lines, } r \neq s, \text{ or } \mathcal{F}_\infty = 4r.$$ 

Now, let us suppose that $\mathcal{F}_\infty = 2r + 2s, r, s$ distinct lines, and $\phi_2 = T\phi_1$, with $\phi_1 \in k[T, X, Y]$. Then $\mathcal{F} : Z^2T^2 + 2ZT^2\phi_1 + \phi_4 = 0$.

To within a linear change of coordinates in $\mathbf{P}^3$, we can assume that $\mathcal{F}_\infty : \{X^2Y^2 = 0 = T\}$ and let

$$\phi_4 = X^2Y^2 + T(ax^3 + bx^2y + cxy^2 + dy^3) + T^2\psi_2, \quad \psi_2 \in k[T, X, Y].$$
The equation for the surface $\mathcal{F} : Z^2T^2 + 2ZT^2\phi_1 + \phi_4 = 0$, can also be written in the form

$$
\mathcal{F} : [Y^2 + T(aX + bY)][X^2 + T(cX + dY)]
+ T^2[Z^2 + 2Z\phi_1 - (aX + bY)(cX + dY) + v_2] = 0.
$$

This means that the line $r : T = X = 0$ and the quartic $Q$, which is the intersection of the quadrics $Y^2 + T(aX + bY) = 0$ and $Z^2 + 2ZT\phi_1 - (aX + bY)(cX + dY) + v_2 = 0$, belong to $\mathcal{F}$. Now, the curves $r$ and $Q$ on $\mathcal{F}$ are skew because

$$
r \cap Q = \{T = X = Y^2 = Z^2 = 0\} = 0.
$$

This fact, from Lemma 1, contradicts the hypothesis that $\mathcal{F}$ is almost-factorial. So, if $T$ divides $\phi_2$, then $\mathcal{F}_\infty = 4r$.

### 4. Quartic Surfaces in $\mathbb{P}^3$ with a Tacnodal Point and Two Distinct Principal Tangents

As a first step in the investigation into almost-factoriality for the quartic surfaces in $\mathbb{P}^3$ with a tacnodal point, we have

**Lemma 5.** Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ be a quartic surface in $\mathbb{P}^3$ with $\mathcal{F} : \{T = 0\} = \mathcal{F}_\infty = 2r + 2s$, $r \neq s$. If we assume that surfaces $\mathcal{G} : G = 0$ and $\mathcal{H} : H = 0$ exist such that $4\deg(\mathcal{G})r = \mathcal{G} \cdot \mathcal{F}$ and $4\deg(\mathcal{H})s = \mathcal{H} \cdot \mathcal{F}$, then the surfaces $\mathcal{G}, \mathcal{H}$ must be quadrics $Q_1 = 0$, $Q_2 = 0$ and $\mathcal{F}$ can be written as

$$
\mathcal{F} : Q_1Q_2 + T^4 = 0.
$$

**Proof.** Let us take $\mathcal{F}$, with $\mathcal{F}_\infty = 2r + 2s$, $r \neq s$. To within a suitable choice of the coordinates in $\mathbb{P}^3$, we can assume that $\mathcal{F}_\infty = \{X^2Y^2 = 0\} \cdot \{T = 0\}$, where $r : X = T = 0$, $s : Y = T = 0$. 
Let $\mathcal{G} : G = 0$ and $\mathcal{H} : H = 0$ be surfaces of degree $n$ such that $4rn = \mathcal{G} \cdot \mathcal{F}$ and $4ns = \mathcal{H} \cdot \mathcal{F}$ (we can assume that $\deg(G) = \deg(H)$, substituting possible $G$ and $H$ with their suitable powers). In the pencil of surfaces $\Phi : \lambda GH + \mu T^{2n} = 0$, we consider $\Phi_0 : \lambda_0 GH + \mu_0 T^{2n} = 0$, which passes through a point $P_0$ on $\mathcal{F} \cap \{T \neq 0\}$. The surface $\Phi_0$ cut on $\mathcal{F} : F = 0$ a divisor that is the sum of $4nr + 4ns$ and of a curve passing through $P_0$, and its degree is strictly greater than $8n = \deg \Phi_0 \deg F$. According to Bézout’s theorem, the surface $\Phi_0 = 0$ is reducible and $\mathcal{F}$ is therefore one of its components. So, a homogeneous polynomial $L \in k[T, X, Y]$ of degree $2n - 4$ exists for which
\[
\lambda_0 GH + \mu_0 T^{2n} = FL.
\]
But the intersection between $T^{2n} = 0$ and $\Phi_0 : \lambda_0 GH + \mu_0 T^{2n} = 0$ is $4nr + 4ns$, and this coincides with the intersection between $T^{2n} = 0$ and $\mathcal{F}$. It follows that $\{L = T = 0\} = 0$, thus $L = c \in k$, $c \neq 0$, $\deg(L) = 8n - 4 = 0$, then $n = 2$. In the light of all the above, $\lambda_0 GH + \mu_0 T^4 = cF$ and we can assume $\lambda_0 = c$, $\mu_0 = c$, so we can write $F = Q_1 Q_2 + T^4$, where
\[
Q_1 = X^2 + T(Z + aX + bY + cT) = 0,
\]
and
\[
Q_2 = Y^2 + T(Z + a'X + b'Y + c'T) = 0.
\]
The coefficients of the monomials $X^2$ and $Y^2$ can both be assumed to be 1.

Instead of $Q_2 = Y^2 + T(Z + a'X + b'Y + c'T) = 0$, we can take $Q_2 = Y^2 + TZ = 0$ by substituting $Z$ with $Z - a'X - b'Y - c'T$ and $(a, b, c)$ with $(a - a', b - b', c - c')$. 
Proposition 3. The quartics $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ in $\mathbb{P}^3$ with $\mathcal{F} : \{T = 0\} = \mathcal{F}_\infty = 2r + 2s$, $r \neq s$, are almost-factorial if and only if they can be written in the form

$$\mathcal{F}(1) : [X^2 + T(Z + aX + bY + cT)][Y^{2} + TZ] + T^4 = 0,$$

with $(4c + b^2 - a^2)^2 = 64$.

$\mathcal{F}(1)$ are 8-almost-factorial.

Proof. From Lemma 5, we can assume that $\mathcal{F} : Q_1Q_2 + T^4 = 0$, where $Q_1 = X^2 + T(Z + aX + bY + cT) = 0$ and $Q_2 = Y^2 + TZ = 0$.

Each of the straight lines $r$, $s$ is a complete intersection of $\mathcal{F}$ with multiplicity 8 because

$$\mathcal{F} \cdot \{Q_1 = 0\} = 8\{T = X = 0\} \text{ and } \mathcal{F} \cdot \{Q_2 = 0\} = 8\{T = Y = 0\}.$$

In addition, we obtain a parametrization of $\mathcal{F}$ on $\mathbb{P}^2$ by means of

$$\tau : (T : X : Y : Z) = (WP_1 : WP_2 : P_3 : WP^2),$$

with the following polynomials of $k[W, U, V] :$

$$P = UW(2V - aW - bW),$$

$$P_1 = -UV^2 - U^2W + bUVW + cUW^2 - W^3,$$

$$P_2 = UV^2 - U^2W - aUVW + cUW^2 - W^3,$$

$$P_3 = UP^2 - WP_1^2 - PW(aP_1 + bP_2 + cP).$$

On $\mathcal{F}$ the transformation

$$(W : U : V) = (T^2 : (X^2 + ZT + T(aX + bY + cT)) : (Y - X)T),$$

is the inverse of $\tau$. To apply Proposition 1 to $\mathcal{F}$, we must compute the polynomial defining the set of non-regularity of the parametrization on $\mathcal{F}$. In the present case, the polynomial is
\[ M = T^7[(a + b)T + 2X - 2Y]^2 \left[ X^2 + TZ + T(aX + bY + cT) \right]^2. \]

\( M = 0 \) intersects \( \mathcal{F} \) along the two lines \( r, s \), and along the section between \( \mathcal{F} \) with the plane 

\[ \pi : (a + b)T + 2X - 2Y = 0. \]

As a result, the intersection between \( \pi \) and \( \mathcal{F} \) generically splits into two conics, \( C_1 \) and \( C_2 \), and they coincide if \( (4c + b^2 - a^2)^2 = 64 \). For 
\[ c = \frac{a^2 - b^2}{4} - 2 \]
and for 
\[ c = \frac{a^2 - b^2}{4} + 2, \]
indeed we have, respectively, the surfaces

\[ \left( \frac{a + b}{2} T + X \right)^2 - T^2 + TZ = 0, \quad \left( \frac{a + b}{2} T + X \right)^2 + T^2 + TZ = 0. \]

On these two surfaces, the plane \( \pi : (a + b)T + 2X - 2Y = 0 \) intersects them along a conic counted twice. We denote with 

\[ \mathcal{F}(1) : [X^2 + T(Z + aX + bY + cT)][Y^2 + TZ] + T^4 = 0, \]

with \( (4c + b^2 - a^2)^2 = 64. \)

\( \mathcal{F}(1) \) is almost-factorial and its index of almost-factoriality is \( \nu = 8. \)

Now, we prove that \( \mathcal{F} \) is not almost-factorial if \( (4c + b^2 - a^2)^2 \neq 64. \)

Let \( d = b - a. \) When \( c \) satisfies the equation \( 2cd^2 + a^3b + 8 = 0, \) the irreducible conics

\[ \mathcal{D} : \begin{cases} 2X + 2Y - dT = 0, \\ ZT + X^2 - dXT = 0, \end{cases} \]

are a subset of \( \mathcal{F}. \) We consider the affine space \( \mathbb{A}^3 = \mathbb{P}^3 \cap \{ T \neq 0 \}. \) and we will have \( \mathcal{F}_a = \mathcal{F} \cap \mathbb{A}^3 \) and \( \mathcal{D}_a = \mathcal{D} \cap \mathbb{A}^3. \) As \( \mathcal{D}(\mathcal{D}_a) \) are irreducible curves, if \( \mathcal{D} \) is a set-theoretic complete intersection of \( \mathcal{F}, \) then \( \mathcal{D}_a \) will be a set-theoretic complete intersection of \( \mathcal{F}_a \) too.
Let \( x = \frac{X}{T}, \ y = \frac{Y}{T}, \ z = \frac{Z}{T} \) be affine coordinates in \( \mathbb{A}^3 \). We have

\[
D_a = \begin{cases} 
    y = -x + \frac{d}{2}, \\
    z = -x^2 + dx.
\end{cases}
\]

Now, we consider De Jonquieres' transformation of \( \mathbb{A}^3 \)

\[
DJ : \{x_1 = x, \ y_1 = y + x - \frac{d}{2}, \ z_1 = z - dx + x^2\},
\]

\( DJ \) transform \( F_a \) and \( D_a \), respectively, on a surface \( G_a \) and on the straight line \( r : \{y_1 = z_1 = 0\} \) on \( G_a \). As \( DJ \) is an isomorphism of \( \mathbb{A}^3 \), \( D_a \) is a complete intersection of \( F_a \) if \( r \) is a complete intersection of \( G_a \). This fact holds if it is satisfied the necessary condition stated in [7] (Lemma 2) analyzing how varies the plane tangent to \( G_a \) at a generic point \((p, 0, 0)\). This is \((d^5 + ad^4 - 16d + 32p)y_1 + (d^4 - 16)z_1 = 0\).

As \((p, 0, 0)\) moves along \( r \), the plane remains fixed if and only if \( d^4 - 16 = 0 \), and it is only in this case that the straight line \( r \) can be a complete intersection on the surface \( G_a \).

Now we compute \( A = (4c + b^2 - a^2)^2 \), substituting \( b = d + a \), \( c = -\frac{d^3b + 8}{2d^2} \) in \( A \). Thus, \( A = 32 + \frac{256}{d^4} + d^4 \) and \( A = 64 \) if and only if \( d^4 = 16 \). This leads us to conclude that, if \((4c + b^2 - a^2)^2 \neq 64 \), then the curves \( D \) on \( F \) cannot be a set-theoretic complete intersection of \( F \). This goes to show that \( F \) is almost-factorial if and only if \((4c + b^2 - a^2)^2 = 64 \).
5. Quartic Surfaces in $\mathbb{P}^3$ with a Tacnodal Point and Only One Principal Tangent

We can write $\mathcal{F} : (ZT + \phi_2)^2 - \Delta = 0$, where $\Delta = \phi_2^2 - \phi_4$.

Let us consider the birational transformation $\tau : \mathbb{P}^3 \longrightarrow \mathbb{P}_1^3$

$$\tau : (T_1 : X_1 : Y_1 : Z_1) = (T^2 : TX : TY : (TZ + \phi_2)), \quad (4)$$

and we have, for its restriction on $\mathcal{F}$

$$\tau^{-1} : (T : X : Y : Z) = (T_1^2 : T_1X_1 : T_1Y_1 : (T_1Z_1 - \phi_2)),$$

where $\phi_2' = \phi_2(T_1, X_1, Y_1)$.

The set of non-regularity of $\tau$ is $T = 0$, and $T_1 = 0$ for $\tau^{-1}$, because we have

$$\tau \circ \tau^{-1} : (T : X : Y : Z) = (T^4 : T^3X : T^3Y : T^3Z),$$

and

$$\tau^{-1} \circ \tau : (T_1 : X_1 : Y_1 : Z_1) = (T_1^4 : T_1^3X_1 : T_1^3Y_1 : T_1^3Z_1).$$

Using $\tau$, we obtain

$$\tau(\mathcal{F}) : [T_1Z_1 - \phi_2(T_1, X_1, Y_1)]T_1^2 + \phi_2(T_1^2, X_1T_1, Y_1T_1))^2 - \Delta(T_1^2, X_1T_1, Y_1T_1)$$

$$= T_1^4[Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1)] = 0.$$

The proper transform of $\mathcal{F}$ by $\tau$ is $\mathcal{H} : Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1) = 0$.

If $\mathcal{F}$ is almost-factorial, then the affine part $\mathcal{F} \cap \{T \neq 0\}$ of $\mathcal{F}$ is almost-factorial too because the exceptional divisor for $\tau$ is the line $\mathcal{F} \cap \{T = 0\}$, counted 4 times. So $\mathcal{H}$ is almost-factorial if its affine part $\mathcal{H} \cap \{T_1 \neq 0\}$ is almost-factorial, and $\mathcal{H}_\infty = \mathcal{H} \cap \{T_1 = 0\}$ is a complete intersection in $\mathbb{P}_1^3$. 
To examine $\mathcal{H}$, we have to distinguish between three cases where

$$\Delta = \phi_2^2 - \phi_4$$

is irreducible, or a factor of $\Delta$ is $T$ or not $T$.

### 5.1. The surfaces $\mathcal{F}$ with $\mathcal{F}_\infty = 4r$ and $\Delta$ are irreducible

First, we examine a surface of the equation $\mathcal{F} : Z^2 T^2 + \phi_4(T, X, Y) = 0$ of the kind $\mathcal{H} : Z^2 T^2 - \Delta(T_1, X_1, Y_1) = 0$ just found.

**Lemma 6.** The polynomial $\phi_4 = -X^4 + T Y^3 + T(X\psi_2 + T \chi_2)$ is irreducible for every homogeneous polynomial $\psi_2, \chi_2 \in k[T, X, Y]$.

**Proof.** A suitable splitting of $\phi_4$ in $k[T, X, Y]$ would be one of two kinds:

$$\phi_4 = [-X^3 + a X^2 Y + b XY^2 + c Y^3 + T(a)][X + d Y + e T]; \quad a, b, c, d, e \in k,$$

$$\phi_4 = [-X^2 + a X Y + b Y^2 + T(a)][X^2 + c XY + d Y^2 + T(a)]; \quad a, b, c, d \in k.$$

As in $\phi_4, X^3 Y, X^2 Y^2, XY^3$ have zero as a coefficient, and in both the equalities it must hold that $a = b = c = d = 0$; but then the monomial $TY^3$ has zero as a coefficient: Contradiction!. This shows that $\phi_4$ is irreducible.

**Lemma 7.** Let $\mathcal{F} : Z^2 T^2 + \phi_4 = 0 \subset \mathbb{P}^3$ be a quartic surface, where

$$\phi_4 = T Y^3 + T[X\psi_2(T, X, Y) + T\chi_2(T, X, Y)] + X^4.$$

A suitable plane $Z = lT$ exists that intersects the surface $\mathcal{F}$ along an irreducible curve $\mathcal{C} : \mathcal{F} \cap \{Z = lT\}$, which proves to be singular in at least one point $P_0 \notin \{T = 0\}$.

By means of a suitable choice of coordinates in $A^3 = \mathbb{P}^3 - \{T = 0\}$, we can assume $\phi_4$ of the form

$$\phi_4 = -(X + a T)^2 X^2 + T Y(b X^2 + c XY + Y^2 + d X T + e Y T),$$

with $a, b, c, d, e \in k$. 
Proof. Let us first prove the existence of a section $C : \{Z - lT = \phi_4 = 0\}$ on $\mathcal{F}$ that is irreducible and it has a singular point $P_0$ on it that does not belong to the plane $T = 0$. Let $F = l^2T^4 - X^4 + T\phi_3$, where $\phi_3 = Y^3 + X\varphi_2(T, X, Y) + T\chi_2(T, X, Y)$; by choosing suitable $l \in k$, the homogeneous system in $T, X, Y$

\[
\begin{align*}
\frac{\partial F}{\partial T} &= 4l^2T^3 + \phi_3 + T\frac{\partial \phi_3}{\partial T} = 0, \\
\frac{\partial F}{\partial X} &= -4X^3 + T\frac{\partial \phi_3}{\partial X} = 0, \\
\frac{\partial F}{\partial Y} &= T\frac{\partial \phi_3}{\partial Y} = T[3Y^2 + X(\cdots) + T(\cdots)] = 0,
\end{align*}
\]

has at least one solution that gives a point $P_0 = (t_0 : x_0 : y_0 : u_0)$, with $t_0 \neq 0$. Indeed, a possible point with $t_0 = 0 (0 : x_0 : y_0 : 0)$ singular for $C$, would satisfy

\[
\{0 = T = X = \phi_3 = Z\} = \{0 = T = X = Y^3 = Z\}.
\]

So $x_0 = y_0 = 0$ and such a point $P_0$ does not exist.

Now $C : \{Z - lT = l^2T^4 - X^4 + T\phi_3 = 0\}$ cannot be reducible because the plane $Z = lT$ is transversal to the cone $l^2T^4 - X^4 + [(Y^3 + X\varphi_2(T, X, Y) + T\chi_2(T, X, Y)] = 0$, which is irreducible because, from Lemma 6, the polynomial $-X^4 + T\chi_2(T\varphi_2 + T\chi_2')$ is irreducible, with $\chi_2' = \chi_2 + l^2T^2$.

To within a change of affine coordinates, we can assume that the plane $Z = lT$ is $Z = 0$, that the singular point on $C$ is $P_0 = O = (1 : 0 : 0 : 0)$, and that one of the lines tangent to $C$ through $P_0$ is $Z = Y = 0$ at the point $A = (1 : -a : 0 : 0)$. These assumptions can be drawn in the light of the fact that $\phi_4 = -X^4 + T\phi_3$, and the coefficient of $Y^3$ in $\phi_3$ is 1. In this chosen frame, the curve $C$ is given by

\[
C : \{Z = 0 = -(X + aT)^2X^2 + TY(bX^2 + cXY + Y^2 + dXT + eYT)\},
\]

with $a, b, c, d, e \in k$. 
Proposition 4. The normal quartic surface of $\mathbb{P}^3$

$$\mathcal{F} : Z^2T^2 + \phi_4 = 0, \quad \phi_4 \in k[T, X, Y], \text{ with } \mathcal{F}_\infty = 4r,$$

is almost-factorial if and only if, to within a suitable linear change of coordinates, it is

$$\mathcal{F}^* : Z^2T^2 - X^4 + Y^3T = 0.$$

$\mathcal{F}^*$ is 12-almost-factorial.

Proof. We can assume that $r = Z_\infty Y_\infty$ and now $\phi_4 = -X^4 + T\phi_3,$

where $\phi_3 = Y^3 + (Xv_2 + TX_2),$ and $v_2, \chi_2 \in k[T, X, Y]$ are zero or homogeneous polynomials of degree 2. The coefficient of $Y^3$ can be assumed to be 1 (it cannot vanish otherwise the generic plane $X = \lambda T$ would intersect $\mathcal{F}$ along the lines $T = X = 0$ counted at least twice, in which case $\mathcal{F}$ would be singular in codimension 1). So, the hypotheses of Lemmas 6 and 7 hold, for which we can assume that

$$\mathcal{F} : Z^2T^2 - (X + aT)^2 X^2 + TY(bX^2 + cXY + Y^2 + dXT + eYT) = 0.$$

Now, let us consider the rational transformation $\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^2$

$$\sigma : (T_2 : X_2 : Y_2 : Z_2) = (T^2 : TX : TY : (ZT - X(X + aT)),$$

which has the following inverse transformation on $\sigma(\mathcal{F})$:

$$\sigma^{-1} : (T : X : Y : Z) = (T^2_2 : T_2X_2 : T_2Y_2 : (Z_2T_2 + X_2(X_2 + aT_2)).$$

As a result,

$$\sigma^{-1} \circ \sigma : (T_2 : X_2 : Y_2 : Z_2) = (T^4_2 : T^3_2X_2 : T^3_2Y_2 : T^3_2Z_2)).$$

From this, the factor of non regularity is $T_2 = 0$, which proves that $\sigma$ is biregular on the affine part $\mathbb{P}^2_2 \cap \{T_2 \neq 0\}$. By means of $\sigma$, the surface $\mathcal{F}$ will be transformed into
To within the factor $T_2^5$, we thus find that the proper transform of $F$ is the cubic (monoid with a double point at $O_1 = (1 : 0 : 0 : 0)$).

$F_3 : T_2(Z_2^3 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2) + Y_2^3 + 2X_2^2Z_2 + bX_2^2Y_2 + cX_2Y_2^2 = 0.$

$Y_2^3 + 2X_2^2Z_2 + bX_2^2Y_2 + cX_2Y_2^2 = 0$ is an irreducible cubic cone that is singular along the line $\{X_2 = Y_2 = 0\}$ and intersects the quadric cone

$$\Gamma : Z_2^3 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2 = 0,$$

along the line $\{Z_2 = Y_2 = 0\}$. As this line is irreducible, it is a set-theoretic complete intersection of $F_3$. Then we can apply Proposition 1 to the surface $F_3$, which is therefore almost-factorial and it must be one of the three almost-factorial surfaces (monoids) classified by Stagnaro (see [13], Theorem on page 143). To within a suitable linear change of coordinates in $\mathbb{P}^3$, these monoids are one of the following kinds:

(I) : $T_2(Y_2^3 + X_2Z_2) + X_2^3 = 0,$

(II) : $T_2(X_2Y_2) - Z_2^3 = 0,$

(III) : $T_2(X_2^3) + X_2Z_2^2 + Y_2^3 = 0.$

A priori, the following cases may occur:

(1) If $\Gamma$ is irreducible ($4ea^2 - d^2 \neq 0$), then $F_3$ will be the surface (I) for which the six lines on $F_3$ passing through $O_1$ must concide. This condition is met if the resultant polynomial, with respect to $Z_2$ (which is now $Z_{2e} \neq \Gamma$) between
\[ Z_2^2 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2 \text{ and } Y_2^3 + 2X_2^2Z_2 + X_2Y_2(bX_2 + cY_2), \]

is a sixth power of one linear form in \( X_2, Y_2 \). But such a resultant is

\[
Y_2[4(ab - d)X_2^5 + (4ac - b^2 - 4e)X_2^4Y_2
+ 2(2a - bc)X_2^3Y_2^2 - (c^2 + 2b)X_2^2Y_2^3 - 2cX_2Y_2^4 - Y_2^5],
\]

which would consequently be divisible by \( Y_2^6 \). This implies that

\[ c = b = a = e = d = 0, \]

and this contradicts \( 4a^2e - d^2 \neq 0 \). So \( \mathcal{F} \) cannot be transformed into the \( \mathcal{F}_3 \) of the kind (I).

(2) The cone \( \Gamma \) is reducible (i.e., \( d^2 = 4ea^2 \)) in two distinct or coincident planes. In the first case, \( \mathcal{F}_3 \) is of the kind (II) and both of the two component planes of the cone must intersect the monoid along three coincident lines. The resultant must then be divisible first by \( Y_2^3 \), and then by the third power of a linear form in \( X_2, Y_2 \). This implies that the resultant is \( Y_2^6 \), so \( b = c = 0 \). \( \Gamma \) is therefore the plane \( Z_2 = 0 \) counted twice. This means that the surface is of the kind (III), with \( a = d = e = 0 \), and the plane \( Z_2 = 0 \) must intersect \( \mathcal{F}_3 \) along a line counted three times. We have \( \mathcal{F}_3 : T_2Z_2^3 + Y_2^3 + 2X_2^2Z_2 = 0 \). To within a change of coordinates in \( \mathbb{P}^3 \), the assigned surfaces

\[ \mathcal{F} : Z^2T^2 + \phi_4 = 0, \quad \phi_4 \in k[T, X, Y], \text{ with } \mathcal{F}_\infty = 4r, \]

are

\[ \mathcal{F}^* : Z^2T^2 - X^4 + TY^3 = 0. \]

This demonstrates Proposition 4. We note that the point \( A = (1 : -a : 0 : 0) \) becomes \( O \), and \( C \) has a triple point at \( O \). \( \mathcal{F}^* \) can be parametrized on the plane \( \mathbb{P}^2 \) by
ALMOST-FACTORIAL QUARTIC SURFACES WITH A ...

\[(T : X : Y : Z) = (W^3(W^2 - U^2)^2 : W^2V^3(W^2 - U^2) : WV^4(W^2 - U^2) : V^6U).\]

The inverse correspondence on \( \mathcal{F}^* \) is \((W : U : V) = (X^2 : TZ : XY)\) with the factor of non-regularity \( M = X^6Y^6T.\)

It is easy to see that the surface \( \mathcal{F}^* \) is 12-almost-factorial. The surface \( \mathcal{F}^* \) was investigated in [6], Example 1, page 290.

**Remark 2.** In the above-mentioned paper [6], it had been shown how computing a surface \( \mathcal{G} \), whose complete intersection with \( \mathcal{F} \) is \( \mathcal{C} \) with multiplicity \( \mu = \deg(\mathcal{F}) \deg(\mathcal{G}) / \deg(\mathcal{C}) \) for every irreducible curve \( \mathcal{C} \) on \( \mathcal{F} \), and some examples were given.

**Proposition 5.** Let \( \mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0 \), be a quartic surface on \( \mathbb{P}^3 \) with \( \mathcal{F} \cdot \{T = 0\} = \mathcal{F}_x = 4\mathfrak{r} \), and \( \Delta = \phi_2^2 - \phi_4 \) is irreducible. \( \mathcal{F} \) is almost-factorial if and only if \( \mathcal{F} \) can be written in the form

\[\mathcal{F}(2) : (ZT + aX^2)^2 - X^4 + Y^3T = 0, \quad a^2 \neq 1.\]

\( \mathcal{F} \) will be 12-almost-factorial.

**Proof.** We have \( \mathcal{F} \cdot \{T = 0\} = 4\mathfrak{r} \), so we can write

\[\phi_4 = X^4 + T\phi_3(T, X, Y), \quad \phi_3 \in k[T, X, Y].\]

Let \( \phi_2 = aX^2 + bXY + cY^2 + T(dx + eY + fT), \quad a, b, c, d, e, f \in k. \)

From the birational transformation (4), the proper transform of \( \mathcal{F} \) by \( \tau \) is \( \mathcal{H} : Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1) = 0 \), where \( \Delta \) is irreducible. So, we can apply Proposition 4 to \( \mathcal{H} \). In particular, in a suitable coordinates’ frame of \( \mathbb{P}_1^3 \), we see that \( \Delta' = \phi_2(T_1, X_1, Y_1)^2 - X_1^4 - T\phi_3(T_1, X_1, Y_1) \) can be written in the form of \(-X_1^4 + Y_1^3T_1. \) Now \( Y_1^4 \) and \( X_1^3Y_1^2 \) do not appear in \( \Delta' \), so it necessarily is \( b = c = 0 \), and therefore, we have \( \phi_2 = aX^2 + T(dx + eY + fT), d, e, f \in k. \) By replacing the coordinates in \( \mathbb{P}^3 \) with
we obtain surfaces of the kind

\[(Z'T' + aX^2)^2 - X^4 + Y^3T' = 0, \quad a^2 \neq 1.\]

We can thus conclude that, to within a change of coordinates, when \(\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 4r\) and \(\Delta = \phi_2^2 - \phi_4\) is irreducible, the only quartic almost-factorial surfaces in \(\mathbb{P}^3\) are of the kind

\[\mathcal{F}(2) : (ZT + aX^2)^2 - X^4 + Y^3T = 0, \quad a^2 \neq 1.\]

\(\mathcal{F}(2)\) are 12-almost-factorial.

5.2. The surfaces \(\mathcal{F}\) with \(\mathcal{F}_\infty = 4r, \Delta\) reducible, \(T\) not factor of \(\Delta\)

**Proposition 6.** Let \(\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0\), be a quartic surface in \(\mathbb{P}^3\) with \(\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 4r\) and \(\Delta = \phi_2^2 - \phi_4\) reducible, but \(T\) does not divide \(\Delta\). In a suitable coordinates’ frame, \(\mathcal{F}\) is almost-factorial if and only if it is of the kind

\[\mathcal{F}(3) : (ZT + aX^2 + Y^2 + T(a_1X + b_1Y))^2\]

\[-[(a-1)X^2 + Y^2 + 2T(dX + eY)][(a+1)X^2 + Y^2 + 2T(fX + gY)] = 0,\]

where \(a^2 \neq 1, (e, g) \neq (0, 0)\), with \((a + f^2/g^2)^2 = 1\), and \(\text{def} \ g \neq 0\). \(\mathcal{F}(3)\) is 4-almost-factorial.

**Proof.** Let \(\phi_2 = aX^2 + 2bXY + cY^2 + T\phi_1\), and now \(\phi_4 = X^4 + T\phi_3\), with \(\phi_1, \phi_3 \in k[T, X, Y]\) suitable polynomials vanishing or of degrees 1 and 3, respectively. If \(\phi_1 = 0 = \phi_3\), \(\mathcal{F}\) has two tacnodal points and we can apply Lemma 3 and Proposition 2 to \(\mathcal{F}\). Such a surface is not almost-factorial. Then we can assume that \((\phi_1, \phi_3)\) does not vanish and, as done in Lemma 3, (5), we can assume that \(c = 1, b = 0, a^2 \neq 1\). Let \(\phi_1 = a_1X + b_1Y + c_1T\). Thus
\[ \Delta(T, X, Y) = \phi_2^2 - \phi_4 = (aX^2 + Y^2 + T\phi_1)^2 - X^4 - T\phi_3, \]

and, since \( \Delta \) is reducible but \( T \) does not divide \( \Delta \) every factor

\[
(aX^2 + Y^2)^2 - X^4 = [(a - 1)X^2 + Y^2][(a + 1)X^2 + Y^2],
\]

is different from zero. So, it must be that \( \Delta = \psi_2\chi_2 \) with non-vanishing suitable polynomials of \( k[T, X, Y] \). Here we can assume that \( O_1 = (1 : 0 : 0 : 0) \in F \) and at \( O_1 \) satisfies \( \phi_1 = \psi_2 = \chi_2 = 0 \) (possibly by substituting \( X, Y \) with suitable \( X' = X + uT, Y' = Y + vT, u, v \in k \)). So, we can suppose \( c_1 = 0 \), so \( \phi_1 = a_1X + b_1Y \), and

\[
\psi_2 = (a - 1)X^2 + Y^2 + 2T(dX + eY),
\]

\[
\chi_2 = (a + 1)X^2 + Y^2 + 2T(fX + gY).
\]

The equation for \( F \) can be written in the form: \( (ZT + \phi_2)^2 = \psi_2\chi_2 \).

We can obtain a parametrization of \( F \) on \( \mathbb{P}^2 \) by assuming first

\[
\frac{V}{W} = \frac{ZT + \phi_2}{\psi_2}, \quad \frac{U}{W} = \frac{Y}{X},
\]

and then, from the relation

\[
\frac{V^2}{W^2} = \frac{(ZT + \phi_2)^2}{\psi_2^2} = \frac{\chi_2}{\psi_2},
\]

we shall have \( V^2\psi_2 = W^2\chi_2 \). So it must be

\[
V^2[(a - 1)X^2 + \frac{U^2}{W^2}X^2 + 2T\frac{U}{W}(dX + e\frac{U}{W}X)]
\]

\[
= W^2[(a + 1)X^2 + \frac{U^2}{W^2}X^2 + 2T\frac{U}{W}(fX + g\frac{U}{W}X)].
\]
Omitting the factor $X$ from the relation above, we shall have

\[
\frac{X}{T} = \frac{WN}{D} \quad \text{and} \quad \frac{Y}{T} = \frac{UN}{D},
\]

where

\[
N = 2[W^2(fW + gU) - V^2(dW + eU)],
\]

\[
D = V^2[(a - 1)W^2 + U^2] - W^2[(1 + a)W^2 + U^2].
\]

Finally, from $W(ZT + \phi_2) = V\psi_2$, we first have

\[
ZT = \frac{V\psi_2(T, X, Y) - W\phi_2(T, X, Y)}{W},
\]

$\phi_2(T, X, Y)$ and $\psi_2(T, X, Y)$ being homogeneous of degree 2, we shall have

\[
\frac{T^2}{D^2} \phi_2\left(\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}\right) = \frac{T^2}{D^2} \phi_2(D, WN, UN),
\]

\[
\frac{T^2}{D^2} \psi_2\left(\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}\right) = \frac{T^2}{D^2} \psi_2(D, WN, UN).
\]

Finally,

\[
\frac{Z}{T} = \frac{P}{WD^2},
\]

where we have put

\[
P = V\psi_2(D, WN, UN) - W\phi_2(D, WN, UN) \in k[U, V, W],
\]

a polynomial of degree 9. Only if it is supposed $N \neq 0$, i.e., $(d, e, f, g) \neq (0, 0, 0, 0)$, we obtain a parametrization for the surface $\mathcal{F}$ given by

\[
\frac{X}{T} = \frac{W^2ND}{WD^2}, \quad \frac{Y}{T} = \frac{WUND}{WD^2}, \quad \frac{Z}{T} = \frac{P}{WD^2}.
\]
and this is a restriction to $\mathcal{F}$ of the rational map $\sigma : \mathbb{P}^2 \to \mathbb{P}^3$


The rational transformation $\mathbb{P}^3 \to \mathbb{P}^2$

$$(W : U : V) = (X \psi_2 : Y \psi_2 : X(ZT + \phi_2)),$$

induces one birational transformation inverse of $\sigma$ on $\mathcal{F}$. When we substitute, mod $\mathcal{F}$, respectively,

$W, U, V$ with $X \psi_2, Y \psi_2, X(ZT + \phi_2)$,

we have, of course $W = X \psi_2, U = Y \psi_2$, and from (5) and (6),

$$D = TX^2 \psi_2^3 R, \quad N = TX^2 \psi_2^3 R,$$

where $R = 2(dX + eY)[(a + 1)X^2 + Y^2] - 2(fX + gY)[(a - 1)X^2 + Y^2]$.

From (7), (8), and (9) results

$$P = V \psi_2(D, WN, UN) - W \phi_2(D, WN, UN)$$

$$= V \psi_2(TX^2 \psi_2^3 R, X^3 \psi_2^3 R, YX^2 \psi_2^3 R) - W \phi_2(TX^2 \psi_2^3 R, X^3 \psi_2^3 R, YX^2 \psi_2^3 R)$$

$$= X^4 \psi_2^6 R^2[V \psi_2(T, X, Y) - W \phi_2(T, X, Y)] = X^4 \psi_2^6 R^2 ZTW = ZTX^5 \psi_2^7 R^2.$$

Therefore, we have for $T, X, Y, Z$, respectively,

$$T(TX^5 \psi_2^7 R^2), \quad X(TX^5 \psi_2^7 R^2), \quad Y(TX^5 \psi_2^7 R^2), \quad Z(TX^5 \psi_2^7 R^2).$$

Thus, the factor of non-regularity of $\sigma$, mod $\mathcal{F}$, is

$$M = TX^5 \psi_2^7 R^2,$$

where $R = 2(dX + eY)[(a + 1)X^2 + Y^2] - 2(fX + gY)[(a - 1)X^2 + Y^2]$.
We note that \( R \) splits into three linear forms in \( X, Y \), and \( R \) is just the resultant of \( \psi_2 \) and \( \chi_2 \) with respect to \( T \).

Now the surface \( M = 0 \) cuts the following curves on \( F \):

- the section of \( F \) with the quadric \( \psi_2 = 0 \); this is the irreducible quartic \( C_4 : \psi_2 = ZT + \phi_2 = 0 \) (counted twice), if \( \psi_2 \) is irreducible; otherwise, if \( \psi_2 \) is reducible, we obtain two conics (counted twice), each of which is the intersection of \( ZT + \phi_2 = 0 \) with a plane component of \( \psi_2 = 0 \);

- \( T = 0 \) gives the line \( r : X = T = 0 \) on \( F \) counted 4 times;

- the section between \( F \) and \( X = 0 \) splits into the line \( r \) and the irreducible (rational) plane cubic

\[
C_3 : \{ X = 0 = Z^2 T + 2 Z Y (Y + b_1 T) + 2 Y^3 (b_1 - e - g) + Y^2 T (b_1^2 - 4 e g) \}.
\]

Now \( r \) is a set-theoretic complete intersection of \( F \) with \( T = 0 \) and also \( C_3 \) because on \( F \), we have

\[
\text{div}_F(-x^4/t) = 4C_3 + 4r - 4r = 4C_3.
\]

If we set \( G = Z^2 T + 2 Z \phi_2 + \phi_3 \) is \( \text{div}_F(g) = 4C_3 \) and therefore \( \mathcal{G} : \{ G = 0 \} \) intersects \( F \) along \( 4C_3 \).

- \( R = 0 \) gives three planes passing through the line \( X = Y = 0 \), each of them, different from \( X = 0, Y = t X \) intersects \( F \) along a reducible quartic if and only if, for \( Y = t X \Delta \) becomes a square \( \Delta = \chi_2 \psi_2 = A^2 X^4 \), with \( A \in k \). We are interested in the planes \( Y = t X \) over which it results \( \overline{R} = 0 \) and so

\[
A^2 X^4
= 4(d - et)(f - gt)X^2 T^2 + 4(f - gt)(a - 1 + t^2)X^3 T + [(a + t^2)^2 - 1]X^4.
\]
When \( t = \frac{f}{g} \) and \( A^2 = (a + \frac{f^2}{g^2})^2 - 1 \neq 0, R = 0 \) implies \( \frac{f}{g} = \frac{d}{e} \). The plane \( fX + gY = 0 \) intersects \( F \) according the two different conics

\[ \{ fX + gY = 0 = ZT + (a_1 - b_1 \frac{f}{g})XT + (\pm A + a + \frac{f^2}{g^2})X^2 \}\].

Arguing similarly the final part of proof of Proposition 3, one shows that these conics can not be complete intersection on \( F \).

When \( t = \frac{f}{g} = \frac{d}{e} \) and \( A^2 = (a + \frac{f^2}{g^2})^2 - 1 = 0 \), i.e., \( a = \pm 1 - \frac{f^2}{g^2} \), we find the two double conics

\[ 2\{ \psi_2 = 0 = ZT + \phi_2 \}, \text{ and } 2\{ \chi_2 = 0 = ZT + \phi_2 \}. \]

The plane \( dX + eY = 0 \), if \( fe \neq gd \), intersects \( F \) according complete intersections: A double conic if \( a = 1 - \frac{d^2}{e^2} \), or an irreducible quartic.

The same is for the plane \( fX + gY = 0 \), if \( fe \neq gd \). It intersects \( F \) according a double conic if \( a = -1 - \frac{f^2}{g^2} \), or an irreducible quartic.

From Proposition 1, \( F \) is almost-factorial if and only if \( (a + \frac{f^2}{g^2})^2 = 1 \), with \( \text{def } g \neq 0 \). \( F \) results 4-almost-factorial.

Let us denote the said almost-factorial surface with

\[ F(3) : (ZT + aX^2 + Y^2 + T(a_1X + b_1Y))^2 \]

\[ - [(a-1)X^2 + Y^2 + 2T(dX + eY)][[(a + 1)X^2 + Y^2 + 2T(fX + gY)] = 0, \]

with \( (a + \frac{f^2}{g^2})^2 = 1 \), \( \text{def } g \neq 0 \).
5.3. Quartic surfaces $\mathcal{F}$ with $\mathcal{F}_\infty = 4r$ and with $T$ dividing $\Delta$

We can write $\Delta = T^2\psi_3(T, X, Y)$, and not $\Delta = T^2\psi_2(T, X, Y)$, because the surface

$$(ZT + \phi_2)^2 - T^2\psi_2(T, X, Y) = 0,$$

would be singular along $\{T = \phi_2 = 0\}$.

In the case under investigation, we obtain, by $\tau$ in (4), from the surface $\mathcal{F} : (ZT + \phi_2)^2 - T\psi_3 = 0$ the cubic surface (double plane)

$$\mathcal{H} : Z^2T_1 - \psi_3(T_1, X_1, Y_1) = 0.$$

If $\mathcal{F}$ is assumed to be almost-factorial, so is $\mathcal{F} \cap \{T \neq 0\}$, and the affine part $\mathcal{H}_a = \mathcal{H} \cap \{T_1 \neq 0\}$ must therefore be almost-factorial. Conversely, if $\mathcal{H}_a$ is almost-factorial, so is $\mathcal{F} \cap \{T \neq 0\}$ because the transformation $\tau$ is biregular on $\mathbb{P}^3 \cap \{T \neq 0\}$, and being $\mathcal{F}_\infty = 4r$, $\mathcal{F}$ is also almost-factorial.

We examine all the possibilities in the following proposition:

**Proposition 7.** Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$, be a normal surface in $\mathbb{P}^3$, with $\mathcal{F}_\infty = 4r$. If $T$ divides $\Delta(T, X, Y) = \phi_2^2 - \phi_4$, $\mathcal{F}$ is almost-factorial if and only if, to within a linear change of affine coordinates in $\mathbb{P}^3 - \{T = 0\}$, $\mathcal{F}$ is one of the following:

$$\begin{align*}
\mathcal{F}(4) & : \ [ZT + (aX + bY)^2]^2 - TXY(X + Y) = 0, \quad 0 \neq a \neq b \neq 0; \\
\mathcal{F}(5) & : \ [ZT + (aX + bY)^2]^2 - TXY(X + T) = 0, \quad a \neq 0, b \neq 0; \\
\mathcal{F}(6) & : \ [ZT + (aX + bY)^2]^2 - TX(XY + T^2) = 0, \quad a \neq 0, b \neq 0; \\
\mathcal{F}(7) & : \ [ZT + (aX + bY)^2]^2 - TX(TY + X^2) = 0, \quad b \neq 0; 
\end{align*}$$
\[
\mathcal{F}(8) : \quad \left[ ZT + (aX + bY)^2 \right]^2 - T(T^2Y + X^2) = 0, \quad b \neq 0;
\]
\[
\mathcal{F}(9) : \quad \left[ ZT + (aX + bY)^2 \right]^2 - T(T^3Y + X^3) = 0, \quad b \neq 0.
\]

The surface \( \mathcal{F}(9) \) is 12-almost-factorial; all the others are 4-almost-factorial.

**Proof.** As \( T \) is a factor of \( \Delta \) (reducible), in every case it must hold that
\[
\mathcal{F}_\Delta : \{ T = \phi_4 = 0 \} = \{ T = \phi_2^2 - \Delta = 0 \} = 2\{ T = \phi_2 = 0 \} = 4r,
\]
with \( r \) a suitable line, that we can assume \( r : \{ T = aX + bY = 0 \} \), with \( a, b \) satisfying the condition to avoid \( \mathcal{F} \) is singular along \( r \). So we have
\[
\phi_2 = (aX + bY)^2 + T\psi_1, \quad \Delta = T\psi_3,
\]
\( \Delta \) being reducible we have to examine the case in which the almost-factorial surface \( \mathcal{F} \) will be transformed into the double plane
\[
\mathcal{H} : Z^2T_1 - \psi_3(T_1, X_1, Y_1) = 0.
\]

The exceptional divisor of \( \tau \) is \( \{ T_1 = \psi_3(T_1, X_1, Y_1) = 0 \} \). This divisor splits into straight lines passing through \( Z_\Delta \) and its affine part \( \mathcal{H}_a = \mathcal{H} \cap \{ T_1 \neq 0 \} \) must be almost-factorial. The affine cubic surface \( \mathcal{H}_a \) is then among those listed in [4]. We are interested only in those that have the projective closure with \( \mathcal{H}_\Delta \) splitting into lines passing through \( Z_\Delta \):

1. they are three distinct lines;
2. two of the three lines coincide; or
3. all three lines coincide.

The double cubic planes with the above said property are the following:
For **case 1** in the list linked to [4], we have only the surface n.14, and by changing \((X_1, Y_1, Z_1)\) with \((-X_1, -Y_1, Z_1)\) is \(\mathcal{H}_4 : Z_1^2 T_1 - X_1 Y_1 (X_1 + Y_1) = 0\); this corresponds to the quartic surface

\[
\mathcal{F}(4) : [ZT + (aX + bY)^2]^2 - TXY(X + Y) = 0, \quad 0 \neq a \neq b \neq 0.
\]

For **case 2**, there are two surfaces in the list

- n.45, assuming \(t = -1\) and exchanging \((X, Y, Z)\) with \((X_1, Z_1, -Y_1)\), we have \(\mathcal{H}_5 : Z_1^2 T_1 - X_1 Y_1 (X_1 + T_1) = 0\), and

- n.64, exchanging \((X_1, Y_1, Z_1)\) with \((-X_1, Z_1, -Y_1)\), we have \(\mathcal{H}_6 : Z_1^2 T_1 - X_1 (X_1 Y_1 + T_1^2) = 0\). The corresponding surfaces are

\[
\mathcal{F}(5) : [ZT + (aX + bY)^2]^2 - TXY(X + T) = 0, \quad a \neq 0, b \neq 0,
\]

\[
\mathcal{F}(6) : [ZT + (aX + bY)^2]^2 - TX(Y^2 + T^2) = 0, \quad a \neq 0, b \neq 0.
\]

Finally, for **case 3**, we have the surfaces \(\mathcal{H}\) in \(P^3_1\) that are almost-factorial because \(\mathcal{H} \cdot \{T_1 = 0\}\) is irreducible. We thus find the surfaces n.72, n.76, and n.78. For the surface n.72, exchanging the coordinates \((X_1, Y_1, Z_1)\) with \((-X_1, Z_1, Y_1)\), we obtain \(\mathcal{H}_7 : Z_1^2 T_1 - X_1 (T_1 Y_1 + X_1^2) = 0\), corresponding to the quartic

\[
\mathcal{F}(7) : [ZT + (aX + bY)^2]^2 - TX(Y^2 + X^2) = 0, \quad b \neq 0.
\]

For n.78, exchanging \((X_1, Y_1, Z_1)\) with \((-X_1, Z_1, -Y_1)\), we obtain \(\mathcal{H}_8 : Z_1^2 T_1 - (T_1^2 Y_1 + X_1^2) = 0\), corresponding to

\[
\mathcal{F}(8) : [ZT + (aX + bY)^2]^2 - T(Y^2 + T^2 + X^3) = 0, \quad b \neq 0.
\]

For n.76, exchanging \((X_1, Y_1, Z_1)\) with \((Z_1 - Y_1, -X_1, Z_1 + Y_1)\), we have \(\mathcal{H}_9 : Z_1^2 T_1 - (T_1 Y_1^2 + X_1^3) = 0\), which gives to the quartic
Now we need to examine each surface $\mathcal{F}(i)$, $4 \leq i \leq 9$, to obtain its suitable parametrization on $\mathbb{P}^2$ and the factor of non-regularity $M$. On each surface $\mathcal{F}_i$, $4 \leq i \leq 8$ every factor of $M$ defines a plane that intersects $\mathcal{F}_i$ according to either a conic \{\(ZT + (aX + bY)^2 = 0\)\} counted twice or the line \{\(aX + bY = 0\)\} with multiplicity 4. According to Proposition 1, all such surfaces are 4-almost-factorial.

For the last surface $\mathcal{F}(9)$, the factor of non-regularity is $T^5X^6$. We have

$$\mathcal{F}(9) \cdot \{T = 0\} = 4r, \quad \mathcal{F}(9) \cdot \{X = 0\} = C_1 + C_2,$$

where $C_1 : \{X = 0 = ZT + b^2Y^2 - YT\}$, $C_2 : \{X = 0 = ZT + b^2Y^2 + YT\}$.

Now it is

$$\mathcal{F}(9) \cap \{ZT + (aX + bY)^2 - YT = 0\} = \begin{cases} ZT + (aX + bY)^2 = YT \\ TX^3 = 0 \end{cases}$$

$$= \{T = 0 = (aX + bY)^2\} + \{X^3 = 0 = ZT + (aX + bY)^2 - YT = 0\} = 2r + 3C_1.$$

On $\mathcal{F}(9)$, we consider the divisor of

$$\frac{(zt + (ax + by)^2 - yt)^2}{t} = \frac{(zt + (ax + by)^2)^2 - 2ty(zt + (ax + by)^2) + t^2y^2}{t} = x^3 - 2y(zt + (ax + by)^2) + 2ty^2.$$  

The cubic surface $S : X^3 - 2Y(ZT + (aX + bY)^2) + 2TY^2 = 0$ gives

$$\mathcal{F}(9) \cdot S = 4r + 6C_1 - 4r = 6C_1.$$

Then we consider $\mathcal{F}(9) \cap \{ZT + (aX + bY)^2 + YT = 0\}$. With a similar calculation, we find
From Proposition 1, the surface $\mathcal{F}(9)$ is almost-factorial, and precisely is $\nu = 12$.

All these results are summarized in the following tables; $\nu = \text{index}$ of almost-factoriality, $M = \text{factor of non-regularity}$ for the parametrization of the surface.

**Almost-factorial quartic with $\mathcal{F}_\infty = 4r$ and $T$ dividing $\Delta$**

<table>
<thead>
<tr>
<th>Equation of the surface</th>
<th>Parametrization $\tau$</th>
<th>$\tau^{-1}$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}(4):$ $[TZ + (aX + bY)^2]^2$ $- TXY(X + Y) = 0$ $0 \neq a \neq b \neq 0$</td>
<td>$T : W^2U^2(W + U)^2$ $X : WU^2V^2(W + U)$ $Y : W^2UV^2(W + U)$ $Z : WUV^2(W + U) - V^4[aU + bW]^2$</td>
<td>$W : TY$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^5X^2Y^2(X + Y)^2$</td>
<td>4</td>
</tr>
<tr>
<td>$\mathcal{F}(5):$ $[TZ + (aX + bY)^2]^2$ $- TXY(X + T) = 0$ $a \neq 0, b = 0$</td>
<td>$T : U^2W^2(W - U)^2$ $X : W^2U^3(W - U)$ $Y : WUV^2(W - U)^2$ $Z : W^2U^2V(W - U)$ $-[aWT^2 + bV^2(W - U)]^2$</td>
<td>$W : T(T + X)$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^7X^2(X + T)^2$</td>
<td>4</td>
</tr>
<tr>
<td>$\mathcal{F}(6):$ $[TZ + (aX + bY)^2]^2$ $- TX(Y + T^2) = 0$ $a \neq 0, b = 0$</td>
<td>$T : W^2U^4$ $X : WU^5$ $Y : W^2U^2(V^2 - WU)$ $Z : WVU^4 - [aU^3 + bW(V^2 - WU)]^2$</td>
<td>$W : T^2$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^7X^4$</td>
<td>4</td>
</tr>
</tbody>
</table>
\[ F(7) : \]
\[
[TZ + (aX + bY)^2]^2
- TX(TY + X^2) = 0
\]
\[ b \neq 0 \]
\[
T : W^6
X : W^4(U^2 - WV)
Y : W^3V(U^2 - WV)
Z : W^5U(U^2 - WV) -
-(U^2 - WV)^2(aW + bV)^2
\]
\[
W : TX
U : TZ + (aX + bY)^2
V : TY
M = T^5X^6
\]

\[ F(8) : \]
\[
[TZ + (aX + bY)^2]^2
- T(T^2Y + X^3) = 0
\]
\[ b \neq 0 \]
\[
T : W^6
X : W^5U
Y : W^3(WV^2 - U^3)
Z : W^5V - [aW^2U + b(WV^2 - U^3)]^2
\]
\[
W : T^2
U : TX
V : ZT + (aX + bY)^2
M = T^{11}
\]

\[ F(9) : \]
\[
[TZ + (aX + bY)^2]^2
- T(T^2Y + X^3) = 0
\]
\[ b \neq 0 \]
\[
T : W^6
X : W^4(U^2 - V^2)
Y : W^3V(U^2 - V^2)
Z : W^5U(U^2 - V^2) -
-(U^2 - V^2)^2(aW + bV)^2
\]
\[
W : TX
U : ZT + (aX + bY)^2
V : TY
M = T^5X^6
\]

**References**


